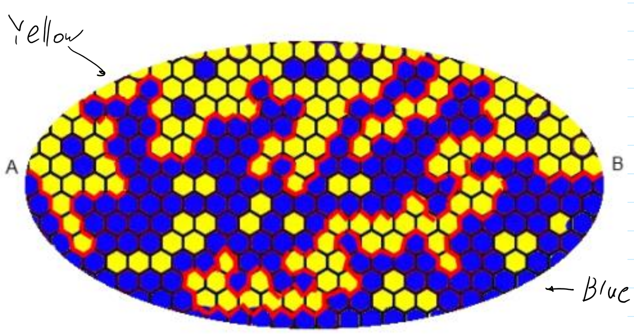
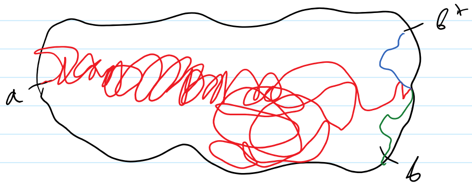
As we discussed in the beginning of the course, SLE_6 is the limit of exploration process



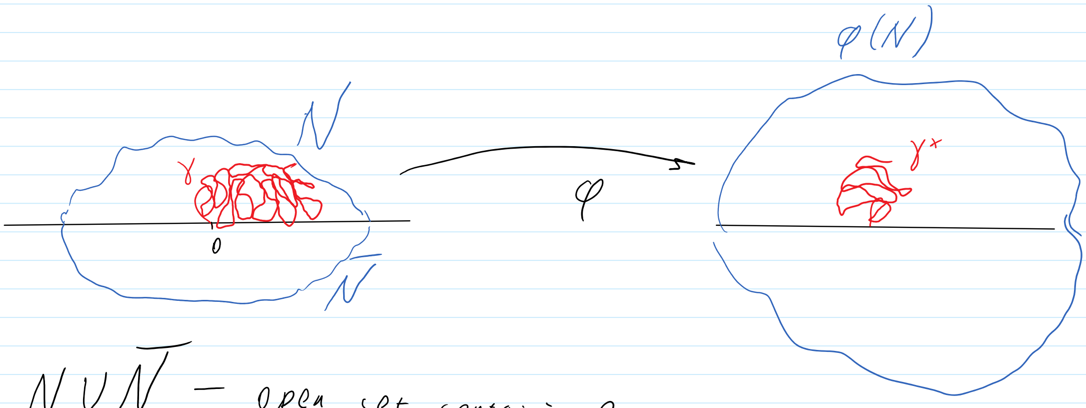
So it should have the Locality property:

Theorem. Let Ω be a domain, $a, b, b^* \in \partial\hat{\Omega}$ - prime ends.
$I :=$ arc between b and b^* in $\partial\hat{\Omega}$.

Let γ be chordal SLE_6 from a to b, $T = \inf\{t : \gamma(t) \in I\}$.
Let γ^* be chordal SLE_6 from a to b^*, $T^* = \inf\{t : \gamma^*(t) \in I\}$.
Then $\{\gamma(t), t \le T\}$ is a time-change of $\{\gamma^*(t), t \le T^*\}$



Setup:



$N \cup \bar{N}$ - open set, containing 0.

γ - SLE_κ from 0 to ∞ $\quad T = \inf\{t : \gamma(t) \notin N \cup \bar{N}\}$.

$\varphi : N \cup \bar{N} \to \mathbb{C}$ - conformal,

$\varphi(N \cap \mathbb{R}) \subset \mathbb{R}$, $\quad \varphi(\bar{z}) = \overline{\varphi(z)}$ - symmetric.

$\gamma^*(t) := \varphi \circ \gamma(t), \quad t < T$

$\Omega_t^* :=$ unbounded component of $\mathbb{H} \setminus \gamma^*[0,t]$.

$g_t^* : \Omega_t^* \to \mathbb{H}$ with hydrodynamic normalization,

$g_t^*(z) = z + \dfrac{a^*(t)}{z} + \dots, \quad z \to \infty$

$a^*(t) = \mathrm{Hcap}\,(\gamma^*[0,t])$

Let $\quad \varphi_t := g_t^* \circ \varphi \circ f_t. \quad (f_t = g_t^{-1})$

$\gamma[0,T]$ is a Löwner curve.

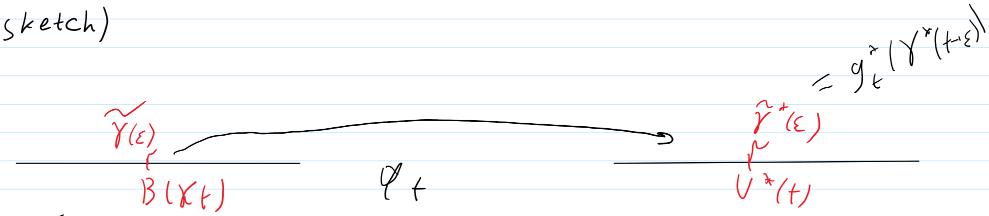
If $\quad \xi^*(t) := g_t^*(\gamma_t^*) = \varphi_t(g_t(\gamma_t)) = \varphi_t(\underbrace{B(\kappa t)}_{SLE_\kappa})$

then, by Löwner,

$$\partial_t g_t^*(z) = \frac{\partial_t a^*(t)}{g_t^*(z) - \xi^*(t)}$$

Claim $\quad \partial_t a^*(t) = 2\left(\varphi_t'(B(\kappa t))\right)^2$

Proof (sketch)



Let $\quad \tilde{\gamma}(\varepsilon) := g_t(\gamma(t+\varepsilon))$ - SLE_κ in $\mathbb{H}$.

$\mathrm{Hcap}\,\tilde{\gamma}(\varepsilon) = 2\varepsilon \quad (SLE_\kappa!)$

$\tilde{\gamma}^*(\varepsilon) := g_t^*(\gamma(t+\varepsilon))$

$\mathrm{Hcap}\,\gamma^*(t+\varepsilon) = \mathrm{Hcap}\,\gamma^*(t) + \mathrm{Hcap}\,\tilde{\gamma}^*(\varepsilon)$ - composition rule

For small enough ε,

$\mathrm{Hcap}\,\tilde{\gamma}^*(\varepsilon) \approx \mathrm{Hcap}\left(\varphi_t'(B(\kappa t))\,\tilde{\gamma}(\varepsilon)\right) \overset{\text{scaling}}{=} \left(\varphi_t'(B(\kappa t))\right)^2 \cdot 2\varepsilon$

(since $\tilde{\gamma}^*(\varepsilon) \approx \varphi_t'(B(\kappa t))\,\tilde{\gamma}(\varepsilon)$

$\mathrm{Hcap}\,\gamma^*(t+\varepsilon) - \mathrm{Hcap}\,\gamma^*(t) = \mathrm{Hcap}\,\tilde{\gamma}^*(\varepsilon)$

Observe that for $z \in N \cup \bar{N} \setminus B(\kappa t)$ we have

$$\partial_t \varphi_t(z) \overset{\text{Chain rule}}{=} \partial_t g_t^*(\varphi \circ f_t(z)) + (g_t^*)'(\varphi \circ f_t(z))' \cdot \partial_t f_t(z) =$$

$$\frac{2\varphi'(B(\kappa t))^2}{\varphi_t(z) - \underbrace{\varphi_t(B(\kappa t))}_{\xi(t)}} + (g_t^*)'(\varphi(f_t(z)))\varphi'(f_t(z)) \cdot \left(-f_t'(z)\frac{2}{z-B(\kappa t)}\right) =$$

$$2\left(\frac{\varphi'(B(\kappa t))^2}{\varphi_t(z) - \varphi_t(B(\kappa t))} - \varphi_t'(z)\frac{1}{z - B(\kappa t)}\right)$$

Take lim when $z \to B(\kappa t)$ to get (requires justification)

$$\boxed{\partial_t \varphi_t(B(\kappa t)) = -3\varphi_t''(B(\kappa t))}$$

So, by Itô, we get

$$d\xi_t^* = d\varphi_t(B(\kappa t)) = \left(\partial_t \varphi_t(B(\kappa t)) + (\kappa/2)\varphi_t''(B(\kappa t))\right)dt + \varphi_t'(B(\kappa t))\, dB_{\kappa t} =$$

$$\left(\frac{\kappa}{2} - 3\right)\varphi_t''(B(\kappa t))dt + \varphi_t'(B(\kappa t))\, dB_{\kappa t}.$$

Consider now $\underline{\kappa = 6}$. $\frac{\kappa}{2} - 3 = 0$, so

$$d\xi_t^* = \varphi_t'(B(6t))\, dB_{6t}.$$

consider time change: $t = \int_0^{r(t)} \varphi_s'(B(6s))^2 ds$

In this time change,

$$d\xi_r^* = d\hat{B}_{\kappa r}, \text{ where } d\hat{B}_{\kappa r} = \varphi_t'(B((6t))\, dB_{\kappa t}.$$

So ξ_r^* is BM scaled by $\sqrt{6}$, $\partial_r a_r^* = 2 \Rightarrow a_r^* = 2r$.

$$\boxed{\partial_r g_r^* = \frac{2}{g_r^* - \xi_r^*}}$$

So we get

<u>Claim</u> $\gamma^*_t \doteq \varphi(\gamma)_t$ is the time change of SLE_6 for $t<T$.

<u>Proof of Theorem.</u> By conformal invariance enough to consider case $\Omega=\mathbb{H}$, $a=0$, $b=\infty$, $b^*=x>0$. (by symmetry).

Let $\varphi(z)=\frac{zx}{z+x}$ — Möbius, $\varphi(\mathbb{R})=\mathbb{R}$, $\varphi(0)=0$, $\varphi(\infty)=x$.

γ - standard SLE_6.

$\varphi(\gamma)$ - SLE_6 in $\mathbb{H}$ from 0 to x.

$N \cup \tilde{N} = \mathbb{C} \setminus [x,\infty)$.

$T=\inf\{t;\ \gamma(t)\in[x,\infty)\}$.

Then $\big(\varphi(\gamma(t))\big)_{t<T}$ has the same law, up to time change, as $\big(\gamma(t)\big)_{t<T}$ $\blacksquare$

Special property of $SLE_{8/3}$.

<u>Theorem</u> (Restriction property).

Let $\tilde{\Omega} \subset \Omega$ - simply connected, $a,b \in \partial\Omega \cap \partial\tilde{\Omega}$ - prime ends.

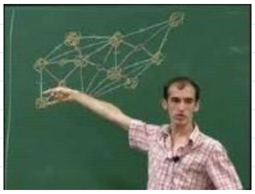
Let γ be $SLE_{8/3}$ in Ω from a to b, $\tilde{\gamma}$ - $SLE_{8/3}$ in $\tilde{\Omega}$.

<u>Then</u> the law of γ conditioned to stay inside $\tilde{\Omega}$ is $\tilde{\gamma}$.

Duality for SLE

Duality for SLE.

Bertrand Duplantier



Julien Dubedat

Theorem (Conjectured by Duplantier; proven by Dubedat; Zhang)
Let $\kappa \geq 4$. Then ∂SLE_κ is a version of $SLE(16/\kappa)$.